Non-Quaternion Number-Systems Containing No Skew Units.

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§1.

In §2 is first given a brief statement of a few important properties of number-systems in general. Next is given a proof of a statement made by Scheffers as to the possibility, in the special class of number-systems here considered, of a selection of units having certain simple multiplicative properties (see p. 371).

In §3 it is shown that the units can be so chosen as to give in general a very much simplified form of multiplication table, and a method is given for deriving systems of the type considered in n units from those in $(n-1)$ units (p. 376).

In §4 is given a theorem on nilfactors (p. 377).

In §5 application of the principles deduced is made to systems the degree of whose characteristic equation is two less than the number of units. Certain general theorems are proved (p. 379), and the systems are reduced to a few typical forms having some peculiar properties (pp. 380, 381, 382).

In §6 the parameters of the systems discussed in §5 are specialized, so far as possible, for the case when the number of units exceeds six, and a table of all the possible non-equivalent forms is given (pp. 385, 386).

§2.

It has been shown by Scheffers* that complex number-systems in n units can be divided into two distinct classes. In any system of the first class, called,

* "Complexe Zahlensysteme," Mathematische Annalen, XXXIX, pp. 306, 310.

after its best-known representative, the quaternion class, there exist three quantities, e_1, e_2, e_3, between which and the modulus, or idemfactor, no linear relation exists, such that

$$\left.\begin{aligned} e_1 e_2 - e_2 e_1 &= 2e_3, \\ e_2 e_3 - e_3 e_2 &= 2e_1, \\ e_3 e_1 - e_1 e_3 &= 2e_2. \end{aligned}\right\} \tag{1}$$

For every number-system of the second class, to which the name non-quaternion is given, it is possible to choose as units quantities

$$u_1 \ldots . u_r \; \eta_1 \ldots . \eta_s$$

which have the following multiplicative properties: $u_i u_j$ and $u_j u_i$, $j \not> i$, are linear functions of $u_1 \ldots . u_{j-1}$. $\eta_i^2 = \eta_i$. $\eta_i \eta_k = 0$, $i \neq k$. $\eta_i u_k$ is zero except for one value of i, say λ_k, when it equals u_k, and similarly $u_k \eta_i$ is zero except for one value of i, say μ_k, when it equals u_k. If $\mu_k \neq \lambda_k$, the unit u_k is said to be skew, otherwise even. This form is called the regular form, and no quaternion system can be put in it, nor does any non-quaternion system contain quantities satisfying the equations (1).

If we consider now non-quaternion systems without skew units, if there be more than one of the quantities η, the system can be reduced to a sum of systems containing each only one η.* Therefore, it is assumed that in the systems here considered there are $(n-1)$ of the units u and only one η, which is the modulus. Any number

$$x = a_1 u_1 + \ldots . a_{n-1} u_{n-1} + \xi\eta$$

(where $a_1 \ldots . a_{n-1}$, ξ are ordinary complex quantities) satisfies the equation†

$$(x - \xi\eta)^v = 0,$$

where $v \not> n$. This is the characteristic equation. If $v = n - \delta$, δ may be called the deficiency of the system.

It follows from this equation that every number σ formed from the units $u_1 \ldots . u_{n-1}$ satisfies the equation $\sigma^{n-\delta} = 0$. It must be possible to choose σ such that $\sigma^{n-\delta-1} \neq 0$, else the characteristic equation would be of lower degree.

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* Ibid., p. 328. † Ibid., p. 316.

The $(n-\delta-1)$ quantities $\sigma, \sigma^2 \ldots . \sigma^{n-\delta-1}$ are all linearly independent, for suppose

$$a_1 \sigma^{n-\delta-1} + a_2 \sigma^{n-\delta-2} + \ldots . = 0.$$

By multiplying enough times by σ we can make all the terms vanish but one, and hence have $\sigma^k = 0$, where $k < n - \delta$, contrary to hypothesis. *Hence, we can use* $\sigma^{n-\delta-1}, \sigma^{n-\delta-2} \ldots . \sigma$ *as* $(n-\delta-1)$ *new units,* $w_1, w_2 \ldots . w_{n-\delta-1}$, *where* $w_a = \sigma^{n-\delta-a}$. The multiplication of the w's is very simple, following from that of the σ's, and will be regular in Scheffers' sense. In fact, $w_i w_k = w_{i+k-n+\delta}$ *or* 0, *according as* $i + k - n + \delta > 0$ *or* $\not> 0$.

These units, with η, make a total of $(n-\delta)$. The remaining δ may be selected from the u's in the following way:

$$\sigma = a_{1,1} u_1 + \ldots . a_{1,n-1} u_{n-1}.$$

From the multiplicative properties of the u's, σ^2 can contain no u of as high an index as occurs in σ, similarly for σ^3 with respect to σ^2, etc. Solve each equation

$$\sigma^k = a_{k,1} u_1 + \ldots .$$

for the u of highest index occurring therein. Each of these u's is, therefore, expressible in a σ^k and u's of lower index, that is, in a w and u's of lower index. Since there are $(n-1)$ u's and only $(n-\delta-1)$ equations, there are δ u's which are not so expressed. These, which will be denoted by $u_{z_1}, u_{z_2} \ldots . u_{z_\delta}$, $z_1 < z_2 \ldots . < z_\delta$, will be taken for the remaining δ units. This is possible, for any u, u_j, other than these, can be expressed, we have seen, in terms of a w, w_k, and u's of lower index than u_j, hence ultimately in terms of $w_1 \ldots . w_k$ and $u_{z_1} \ldots . u_{z_a}$, where u_{z_a} is that one of $u_{z_1}, u_{z_2} \ldots . u_{z_\delta}$ immediately preceding u_j.

The multiplication of the w's we have remarked to be regular. *If we write the units* $w_1 \ldots . w_{n-\delta-1}, u_{z_1} \ldots . u_{z_\delta}$ *in such order that no* w *preceding* u_{z_a} *contains a* u *of as high index as* z_a, *and every* w *following* u_{z_a} *contains a* u *of higher index than* z_a, *which is evidently possible, the multiplication of the new units will be entirely regular.* For if we consider $u_{z_a} w_k$ and $w_k u_{z_a}$, where w_k lies beyond u_{z_a}, the product expressed in u's can contain no u of higher index than $(z_a - 1)$, and the product is accordingly expressible in w's and u_z's preceding u_{z_a}. Similarly for $u_{z_a} w_k$ and $w_k u_{z_a}$, where w_k occurs before u_{z_a}, and also for $u_{z_a} u_{z_\beta}$.

This verifies a statement made without proof by Scheffers,* in considering the cases $\delta = 1$ and $\delta = 2$, that the δ units in addition to $w_1 \dots . w_{n-\delta-1}$, η could be so chosen as to make the table regular, although their position relative to the w's is unknown.

§3.

It is now proposed to show that by abandoning in part the regular form, the multiplication table can be simplified in certain cases. The methods are an extension of those used by Scheffers in considering the case $\delta = 1$.

The units $w_1 \dots . w_{n-\delta-1}$, $u_{z_1} \dots . u_{z_\delta}$ being regular in some order, in which the order of $u_{z_1} \dots . u_{z_\delta}$ is, however, unchanged, u_{z_δ} occurs in none of their products. Therefore, the system $w_1 \dots . w_{n-\delta-1}, u_{z_1} \dots . u_{z_{\delta-1}}, \eta$ is unchanged by the deletion of u_{z_δ}, and is a system of $(n-1)$ units. Since $w_{n-\delta-1} = \sigma$ is in this system, the characteristic equation is the same as before; hence this system of $(n-1)$ units is of deficiency $(\delta - 1)$. It will now be assumed that any system of deficiency $(\delta - 1)$ can, by replacing u_{z_β} by

$$\tau_\beta = u_{z_\beta} + a_{2,\beta} w_2 + \dots . a_{(n-\delta-1),\beta} w_{n-\delta-1}$$
$$\beta = 1, 2 \dots . (\delta - 1)$$

be put in a form having the following properties: $\tau_\beta w_{n-\delta-k}$ and $w_{n-\delta-k} \tau_\beta$ are zero if $k > \beta$, contain only w_1 if $k = \beta$, while if $k < \beta$ they are linear functions of $w_1 \dots . w_{\beta-k+1}, \tau_1 \dots . \tau_{\beta-k}$. We wish to prove that the same can be done for systems of deficiency δ. By the notation $(x, y, z \dots .)$ will be meant a linear function of $x, y, z \dots . .$

Now,

$$u_{z_\delta} w_{n-\delta-1} = u_{z_\delta}(u_1, \dots . u_{n-1}) = (u_1, u_2 \dots . u_{z_\delta - 1}).$$

All of these last u's are expressible in w's and $u_{z_1}, u_{z_2} \dots . u_{z_{\delta-1}}$ (see p, 371), hence, in w's and $\tau_1, \tau_2 \dots . \tau_{\delta-1}$. So

$$u_{z_\delta} w_{n-\delta-1} = c_1 w_1 + \dots . c_{n-\delta-1} w_{n-\delta-1} + d_1 \tau_1 + \dots . d_{\delta-1} \tau_{\delta-1} \tag{2}$$

or

$$u_{z_\delta} \sigma = c_1 \sigma^{n-\delta-1} + \dots . c_{n-\delta-1} \sigma + d_1 \tau_1 + \dots . d_{\delta-1} \tau_{\delta-1}. \tag{3}$$

Similarly,

$$w_{n-\delta-1} u_{z_\delta} = c'_1 w_1 + \dots . c'_{n-\delta-1} w_{n-\delta-1} + d'_1 \tau_1 + \dots . d'_{\delta-1} \tau_{\delta-1}. \tag{2'}$$

$$\sigma u_{z_\delta} = c'_1 \sigma^{n-\delta-1} + \dots . c'_{n-\delta-1} \sigma + d'_1 \tau_1 + \dots . d'_{\delta-1} \tau_{\delta-1}. \tag{3'}$$

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* Ibid, pp. 333, 340, 341.

Case I. $n \not> 2\delta + 1$.

Let $\tau_\delta = u_{z_\delta} + \lambda_2 w_2 + \ldots . \lambda_{n-\delta-1} w_{n-\delta-1}$ where $\lambda_2 \ldots \lambda_{n-\delta-1}$ are arbitrary. Then,

$$\tau_\delta w_{n-\delta-1} = u_{z_\delta} w_{n-\delta-1} + \lambda_2 w_1 + \ldots . \lambda_{n-\delta-1} w_{n-\delta-2} = (c_1 + \lambda_2) w_1$$
$$+ \ldots . (c_{n-\delta-2} + \lambda_{n-\delta-1}) w_{n-\delta-2} + c_{n-\delta-1} w_{n-\delta-1} + d_1 \tau_1 + \ldots . d_{\delta-1} \tau_{\delta-1}. \quad (4)$$

Multiply this by σ^{k-1}, obtaining

$$\tau_\delta w_{n-\delta-k} = e_1 w_1 + \ldots . c_{n-\delta-1} w_{n-\delta-k} + (d_1 \tau_1 \sigma^{k-1} + \ldots . d_{\delta-1} \tau_{\delta-1} \sigma^{k-1}).$$

Now by the law assumed for the multiplication of $\tau_1 \ldots . \tau_{\delta-1}$ with the w's, no higher w can occur from the parenthesis than $w_{\delta-k+1}$ and no higher τ than $\tau_{\delta-k}$. And since $n \not> 2\delta + 1$, $n - \delta - k \not> \delta - k + 1$, and accordingly $\tau_\delta w_{n-\delta-k}$ has as its highest w, $w_{\delta-k+1}$, and as its highest τ, $\tau_{\delta-k}$. If $k > \delta$ the terms all vanish. A similar proof applies to $w_{n-\delta-k}\tau_\delta$. Hence the multiplication of τ_δ with the w's is according to the law assumed for $\tau_1 \ldots . \tau_{\delta-1}$.

From (2′)

$$w_{n-\delta-1} \tau_\delta = (c_1' + \lambda_2) w_1 + \ldots . (c_{n-\delta-2}' + \lambda_{n-\delta-1}) w_{n-\delta-2}$$
$$+ c_{n-\delta-1}' w_{n-\delta-1} + d_1' \tau_1 + \ldots . d_{\delta-1}' \tau_{\delta-1}. \quad (4')$$

Hence, the λ's can be so chosen as to make the coefficient of w_i occurring in $\tau_\delta w_{n-\delta-1}$ the negative of that occurring in $w_{n-\delta-1}\tau_\delta$, except for $i = n - \delta - 1$.

Case II. $n > 2\delta + 1$.

Multiply (3) by σ^δ. Since $n - \delta > \delta + 1$, $\sigma^{\delta+1} \neq 0$. We have then

$$u_{z_\delta} \sigma^{\delta+1} = c_{\delta+1} \sigma^{n-\delta-1} + \ldots . c_{n-\delta-1} \sigma^{\delta+1} + (d_1 \tau_1 + \ldots . d_{\delta-1} \tau_{\delta-1}) \sigma^\delta. \quad (5)$$

But by the assumed law of multiplication the last term is zero. Multiplying by σ^{i-1},

$$u_{z_\delta} \sigma^{\delta+i} = c_{\delta+i} \sigma^{n-\delta-1} + \ldots \ldots c_{n-\delta-1} \sigma^{\delta+i}.$$

So from (5),

$$u_{z_\delta}^2 \sigma^{\delta+1} = g_1 \sigma^{n-\delta-1} + \ldots . g_{n-\delta-2} \sigma^{\delta+1} + c_{n-\delta-1}^2 \sigma^{\delta+1}. \quad (6)$$

But

$$u_{z_\delta}^2 = (u_1 \ldots . u_{z_\delta - 1}) = (w_1 \ldots . w_{n-\delta-1}, u_{z_1} \ldots . u_{z_{\delta-1}})$$
$$= (\sigma, \sigma^2 \ldots . \sigma^{n-\delta-1}, \tau_1 \ldots . \tau_{\delta-1}).$$

Multiplying by $\sigma^{\delta+1}$, since $\tau_1 \ldots . \tau_{\delta-1}$ all make zero into $\sigma^{\delta+1}$, we find that $u^2_{z_\delta}\sigma^{\delta+1}$ does not contain $\sigma^{\delta+1}$, hence, comparing with (6), $c_{n-\delta-1}=0$.

Therefore, equation (3) becomes

$$u_{z_\delta}\sigma = c_1\sigma^{n-\delta-1} + \ldots . c_{n-\delta-2}\sigma^2 + d_1\tau_1 + \ldots . d_{\delta-1}\tau_{\delta-1}. \tag{7}$$

Similarly,

$$\sigma u_{z_\delta} = c_1'\sigma^{n-\delta-1} + \ldots . c_{n-\delta-2}'\sigma^2 + d_1'\tau_1 + \ldots . d_{\delta-1}'\tau_{\delta-1}. \tag{7'}$$

If $n = 2\delta + 2$ the method of Case I can now be followed. If $n > 2\delta + 2$, $n - \delta > \delta + 2$, and $\sigma^{\delta+2} \neq 0$. Multiplying (7) by σ^δ left-handed and (7′) by $\sigma^{\delta-1}$ left-handed and σ right-handed, and equating,

$$c_{\delta+1}\sigma^{n-\delta-1} + \ldots . c_{n-\delta-2}\sigma^{\delta+2} = c_{\delta+1}'\sigma^{n-\delta-1} + \ldots . c_{n-\delta-2}'\sigma^{\delta+2} + d_{\delta-1}'\sigma^{\delta-1}\tau_{\delta-1}\sigma.$$

But $\sigma^{\delta-1}\tau_{\delta-1}$ can contain only w_1 or $\sigma^{n-\delta-1}$. Hence the last term vanishes. Since the σ's are linearly independent, we must, therefore, have

$$c_{\delta+1}' = c_{\delta+1}, \ldots . c_{n-\delta-2}' = c_{n-\delta-2}.$$

Accordingly, if we substitute

$$\tau_\delta = u_{z_\delta} + \lambda_1\sigma^{n-\delta-2} + \ldots . \lambda_\delta\sigma^{n-2\delta-1} - c_{\delta+1}\sigma^{n-2\delta-2} - \ldots . - c_{n-\delta-2}\sigma,$$

where the λ's are arbitrary, there results

$$\tau_\delta w_{n-\delta-1} = (c_1+\lambda_1)w_1 + \ldots . (c_\delta+\lambda_\delta)w_\delta + d_1\tau_1 + \ldots . d_{\delta-1}\tau_{\delta-1}, \tag{8}$$

and likewise

$$w_{n-\delta-1}\tau_\delta = (c_1'+\lambda_1)w_1 + \ldots . (c_\delta'+\lambda_\delta)w_\delta + d_1'\tau_1 + \ldots . d_{\delta-1}'\tau_{\delta-1}. \tag{8'}$$

Multiplying (8) by $\sigma^{k-1} = w_{n-\delta-k+1}$,

$$\tau_\delta w_{n-\delta-k} = (c_k+\lambda_k)w_1 + \ldots . (c_\delta+\lambda_\delta)w_{\delta-k+1} + (d_1\tau_1 + \ldots . d_{\delta-1}\tau_{\delta-1})w_{n-\delta-k+1}.$$

But, by the multiplicative properties of $\tau_1 \ldots . \tau_{\delta-1}$, the highest w occurring from the parenthesis is $w_{\delta-k+1}$, and the highest τ is $\tau_{\delta-k}$. Hence, $\tau_\delta w_{n-\delta-k}$ and, similarly, $w_{n-\delta-k}\tau_\delta$, conform to the given law.

Evidently in (8) and (8′) we can so choose $\lambda_1 \ldots . \lambda_\delta$ that the coefficients of w_i occurring in (8) are the negative of those in (8′). We saw that this could be done for Case I, $n \not> 2\delta + 1$, except for $w_{n-\delta-1}$, should it occur. In the present case it cannot occur, for since $n > 2\delta + 1$, $\delta < n - \delta - 1$.

It has, therefore, been proved that if a system of deficiency $(\delta - 1)$ can be put in a form having certain multiplicative properties, one of deficiency δ can. But, by going through this demonstration with $\delta = 1$, it will be seen that a system of deficiency 1 can so be put, as, indeed, has been shown by Scheffers. Hence, the theorem is true for $\delta = 2, 3 \ldots$.

Consider $\tau_\alpha \tau_\beta$, where $\beta \not> \alpha$.

$$\tau_\alpha \tau_\beta = (u_{z\alpha} + a_2 w_2 + \ldots a_{n-\delta-1} w_{n-\delta-1}) \tau_\beta$$
$$= u_{z_\alpha} \tau_\beta + x$$

where x can contain only

$$w_1 \ldots w_\beta, \ \tau_1 \ldots \tau_{\beta-1}.$$
$$u_{z_\alpha} \tau_\beta = u_{z_\alpha} (u_{z_\beta} + b_2 w_2 + \ldots b_{n-\delta-1} w_{n-\delta-1})$$
$$= u_{z_\alpha} (u_1, u_2 \ldots u_{n-1}) = (u_1 \ldots u_{z_\alpha - 1})$$
$$= (w_1 \ldots w_{n-\delta-1}, \ u_{z_1} \ldots u_{z_{\alpha-1}}).$$

Hence,

$$\tau_\alpha \tau_\beta = e_1 w_1 + \ldots e_\alpha w_\alpha + e_{\alpha+1} w_{\alpha+1} + \ldots e_{n-\delta-1} w_{n-\delta-1} + b_1 \tau_1 + \ldots b_{\alpha-1} \tau_{\alpha-1},$$

or

$$\tau_\alpha \tau_\beta = e_1 \sigma^{n-\delta-1} + \ldots e_\alpha \sigma^{n-\delta-\alpha} + e_{\alpha+1} \sigma^{n-\delta-\alpha-1}$$
$$+ \ldots e_{n-\delta-1} \sigma + b_1 \tau_1 + \ldots b_{\alpha-1} \tau_{\alpha-1}. \quad (9)$$

If $n - \delta \not> \alpha + 1$, $n - \delta - 1 \not> \alpha$, and $e_m \, (m > \alpha)$ does not occur in (9). If $n - \delta > \alpha + 1$, $\sigma^{\alpha+1} \neq 0$, and multiplying (9) by σ^α, since

$$\tau_1 \sigma^\alpha = \tau_2 \sigma^\alpha = \ldots \tau_{\alpha-1} \sigma^\alpha = 0,$$

we find

$$\tau_\alpha \tau_\beta \sigma^\alpha = e_{\alpha+1} \sigma^{n-\delta-1} + \ldots e_{n-\delta-1} \sigma^{\alpha+1}. \quad (10)$$

If $\beta < \alpha$, $\tau_\beta \sigma^\alpha = 0$, and the preceding equation necessitates

$$e_{\alpha+1} = e_{\alpha+2} = \ldots e_{n-\delta-1} = 0.$$

Similarly for $\tau_\beta \tau_\alpha$. If, however, $\beta = \alpha$, (10) becomes

$$\tau_\alpha^2 \sigma^\alpha = e_{\alpha+1} \sigma^{n-\delta-1} + \ldots + e_{n-\delta-1} \sigma^{\alpha+1}. \quad (11)$$

But $\quad \tau_\alpha \sigma^\alpha = p w_1 = p \sigma^{n-\delta-1}.$

So $\quad \tau_\alpha^2 \sigma^\alpha = p \tau_\alpha \sigma^{n-\delta-1}.$

And since $n - \delta > \alpha + 1$, $n - \delta - 1 > \alpha$, so $\tau_\alpha \sigma^{n-\delta-1} = 0$. Therefore, $\tau_\alpha^2 \sigma^\alpha = 0$, and comparing with (11),

$$e_{\alpha+1} = e_{\alpha+2} = \ldots . e_{n-\delta-1} = 0$$

as before.

Therefore, the following facts have been proved: *Every irreducible non-quaternion system without skew units can have selected as its units quantities* $w_1 \ldots . w_{n-\delta-1}$, $\tau_1 \ldots . \tau_\delta$, η *which have the multiplicative properties given below:*

$w_i w_k = w_{i+k-n+\delta}$ *or* 0 *according as* $i + k - n + \delta > 0$ *or* $i + k - n + \delta \not> 0$. $\tau_\alpha w_{n-\delta-k}$ *and* $w_{n-\delta-k} \tau_\alpha$ *are zero if* $k > \alpha$, *contain only* w_1 *if* $k = \alpha$, *while if* $k < \alpha$ *they are linear in* $w_1 \ldots . w_{\alpha-k+1}$, $\tau_1 \ldots . \tau_{\alpha-k}$. *The coefficient of the term in* w_i *occurring in* $\tau_\alpha w_{n-\delta-1}$ *is the negative of that in* $w_{n-\delta-1} \tau_\alpha$ *except for* $i = n - \delta - 1$. $\tau_\alpha \tau_\beta$ *and* $\tau_\beta \tau_\alpha$ $(\beta \not> \alpha)$ *are linear functions of* $w_1 \ldots . w_\alpha$, $\tau_1 \ldots . \tau_{\alpha-1}$.

It will be seen, therefore, that the multiplication of the τ's with each other is not regular, but the remaining multiplicative properties are much simplified from Scheffers' general regular form. If δ should be large compared with $(n - \delta - 1)$ the w-τ form will not be so simple as the w-u_z form, which is regular, but if δ is small compared with $(n - \delta - 1)$ it will be much simpler. By application of the associative law it is easily seen that for $\delta = 2$ the w-τ form is regular if we place the unit τ_1 before $w_{n-\delta-1}$.

If the row and column τ_δ be deleted we have a table in $w_1 \ldots . w_{n-\delta-1}$, $\tau_1 \ldots . \tau_{\delta-1}$, η or $(n - 1)$ units, of deficiency $(\delta - 1)$, which nowhere contains τ_δ. Hence, *if we take every independent system of the type considered in* $(n - 1)$ *units of deficiency* $(\delta - 1)$ *in the* w-τ *form, and border it with a row and column* τ_δ *having the multiplicative properties given above, we shall obtain every possible system in* n *units of deficiency* δ.

This bordering will introduce certain parameters, which can be reduced in number by application of the associative law and also by the fact that $\tau_\delta^{n-\delta} = 0$. Further reductions may be made by introducing new units of such a type as not to change the multiplicative properties. Allowable substitutions are

$$\begin{aligned} \tau_\alpha' &= a_\alpha \tau_\alpha + b_\alpha \tau_{\alpha-1} + \ldots . p_\alpha \tau_1, \\ \sigma' &= a\sigma + b\sigma^2 + \ldots . p\sigma^{n-\delta-1}. \end{aligned}$$

Other substitutions may be available in special cases. If $n > 3\delta$, applications of the associative law can be made irrespective of n. If, however, $n \ngtr 3\delta$, each value of n has to be considered separately.

It will be noticed that with a given original set of units and with σ once chosen the τ's were determined uniquely. The new τ's indicated by the first of the preceding equations are those which could arise from the same choice of σ, but with the given system in a different form, linearly connected with the old form.

It may be asked, conversely, are all systems in the w-τ form non-quaternion systems of the type considered? If they are non-quaternion, they are of that type, from the form of the characteristic equation. It is easy to show from the multiplicative properties of the w-τ form that it is impossible to find three quantities, e_1, e_2, e_3, satisfying equations (1). Hence, no quaternion system can be put in that form.

§4.

A quantity ν such that

$$\nu x = x\nu = 0$$

for all values of x may be called a nilfactor. Such cannot exists in a system containing a modulus. The following theorem will be needed in §6, by incomplete systems being meant systems without a modulus: *If two incomplete systems are equivalent, and the same nilfactor is a unit in each, then if this nilfactor is deleted from each system the deleted systems are equivalent.*

Let the systems be

$$u_1\, u_2 \ldots\ldots u_{n-1}\, \nu,$$
$$u'_1\, u'_2 \ldots\ldots u'_{n-1}\, \nu',$$

where

$$\left.\begin{aligned} u'_i &= a_{i,1}\, u_1 + \ldots\ldots a_{i,n-1}\, u_{n-1} + b_i \nu, \quad i = 1, 2 \ldots\ldots n-1 \\ \nu' &= \nu, \end{aligned}\right\} \tag{1}$$

$$\left.\begin{aligned} u_i &= a'_{i,1}\, u'_1 + \ldots\ldots a'_{i,n-1}\, u'_{n-1} + b'_i \nu', \quad i = 1, 2 \ldots\ldots n-1 \\ \nu &= \nu'. \end{aligned}\right\} \tag{2}$$

Then the systems obtained by deleting ν and ν' are equivalent. For, consider the system

$$v'_i = a_{i,1}\, v_1 + \ldots\ldots a_{i,n-1}\, v_{n-1}, \qquad i = 1, 2 \ldots\ldots n-1 \tag{3}$$

the system $v_1 \dots v_{n-1}$ having the multiplication table of $u_1 \dots u_{n-1}$ with ν deleted. Now, equations (2) can be obtained from equations (1) by substituting ν' for ν in the first $(n-1)$ equations of (1) and then eliminating for the quantities $u_1 \dots u_{n-1}$. Evidently the coefficient of u'_j occurring in u_i of (2) will be unaffected by the values of $b_1 \dots b_{n-1}$ in (1), hence will be the same if $b_1 \dots b_{n-1}$ are zero. Therefore, we obtain from (3)

$$v_i = a'_{i,1} v'_1 + \dots a'_{i,n-1} v'_{n-1}, \qquad i = 1, 2 \dots n-1. \quad (4)$$

ν being a nilfactor,

$$u'_i u'_j = (a_{i,1} u_1 + \dots a_{i,n-1} u_{n-1})(a_{j,1} u_1 + \dots a_{j,n-1} u_{n-1}).$$

Now the multiplicative properties of the u's after deletion are the same as before, except that ν is dropped out. Hence, $v'_i v'_j$ expressed in v's is identical with $u'_i u'_j$ expressed in u's (with $v_k = u_k$), except for the terms in ν. Therefore, from (2) and (4), $v'_i v'_j$ expressed in v''s is identical with $u'_i u'_j$ expressed in u''s (if we set $v'_k = u'_k$), except for terms in ν'. Hence, the multiplication table of the v''s is identical with that of the u''s with ν' deleted, or the deleted u' table can be obtained from the deleted u table by

$$u'_i = a_{i,1} u_1 + \dots a_{i,n-1} u_{n-1}, \qquad i = 1, 2 \dots n-1$$

which proves the theorem.

This proof evidently holds if, instead of having $\nu' = \nu$, we have $\nu' = c\nu$. The theorem is also true if $\nu' = c\nu + x$, $(c \neq 0)$, provided one of ν and ν', say ν, occurs nowhere in the products $u_i u_j$. For if we substitute for ν a new unit $\nu'' = c\nu + x$, then $\nu' = \nu''$, and ν' and ν'' can be deleted. But since ν does not occur in $u_i u_j$, no change is made in the table, and hence the deletion of ν'' has the same effect as that of ν.

§5.

An application of the principles obtained will now be made to the case $\delta = 2$. $n > 2$, else the characteristic equation is of degree lower than unity, which is impossible. If $n = 3$, the characteristic equation becomes $x - \xi\eta = 0$ or $x = \xi\eta$, or there is only one unit in the system, η, not three. The cases $n = 4$, $n = 5$ have been already considered by Scheffers by different methods. These will, therefore, not be considered here except for comparison with certain results true for $n > 5$.

n being greater than five, the general multiplicative properties of the $w - \tau$ form give us the following table, η being omitted as unnecessary:

	w_1	$w_2 \ldots\ldots w_{n-5}$	w_{n-4}	w_{n-3}	τ_1	τ_2
w_1	0	$0 \ldots\ldots 0$	0	0	0	0
w_2	0	$0 \ldots\ldots 0$	0	w_1	0	0
$\vdots$	$\vdots$	$\vdots \quad \vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
w_{n-5}	0	$0 \ldots\ldots w_{n-8}$	w_{n-7}	w_{n-6}	0	0
w_{n-4}	0	$0 \ldots\ldots w_{n-7}$	w_{n-6}	w_{n-5}	0	lw_1
w_{n-3}	0	$w_1 \ldots w_{n-6}$	w_{n-5}	w_{n-4}	aw_1	$c\tau_1 + ew_1 + kw_2$
τ_1	0	$0 \ldots\ldots 0$	0	$-aw_1$	bw_1	$r\tau_1 + fw_1 + pw_2$
τ_2	0	$0 \ldots\ldots 0$	mw_1	$d\tau_1 - ew_1 - kw_2$	$s\tau_1 + gw_1 + qw_2$	$j\tau_1 + hw_1 + iw_2$

By comparison of $(\sigma\tau_2)\,\sigma$ with $\sigma\,(\tau_2\sigma)$, it follows that $k = \frac{a\,(c+d)}{2}$, whence

$$l = \frac{a}{2}\,(3c + d), \quad m = -\,\frac{a}{2}\,(3d + c).$$

By comparing $\tau_2^2\tau_1$ with $\tau_2\,(\tau_2\tau_1)$ and $\tau_1\tau_2^2$ with $(\tau_1\tau_2)\,\tau_2$, there results $r = s = 0$. It is therefore evident that *the table will be regular in Scheffers' sense by placing* τ_1 *before* w_{n-3}; *also* w_1 *is a nilfactor.* These facts are true for $n = 5$, and, if σ is chosen suitably, for $n = 4$.

Further application of the associative law must be made separately for $n = 6$ and $n > 6$. An additional property common to both cases is found to be $p = q = 0$. This gives rise to the important fact that w_1 is unique; that is, *if two systems*

$$w_1 \ldots\ldots w_{n-3},\ \tau_1,\ \tau_2,\ \eta,\ \text{and}\ \ w_1' \ldots\ldots w_{n-3}',\ \tau_1',\ \tau_2',\ \eta$$

are equivalent, then w_1' *equals* w_1 *except for a constant factor.* For

$$\sigma' = A_1\sigma + \ldots\ldots A_{n-3}\sigma^{n-3} + B_1\tau_1 + B_2\tau_2 + C\eta.$$

C must be zero, else $\sigma'^{n-2} \neq 0$. Therefore, σ'^2 can contain only $\sigma^2 \ldots \sigma^{n-3}$, τ_1. For the products of the σ's can contain only $\sigma^2 \ldots . \sigma^{n-3}$, while the products of the τ's with each other and with the σ's can only contain w_1, w_2, τ_1, or σ^{n-3}, σ^{n-4}, τ_1; since $n > 5$, $n - 4 > 1$. Also $n - 3 > 2$, hence neither σ' nor σ'^2 can be w_1'.

Similarly, σ'^3 can contain only $\sigma^3 \ldots . \sigma^{n-3}$, for $\tau_1 \sigma^k = \sigma^k \tau_1 = 0$, $k > 1$, $\tau_2 \sigma^k = \sigma^k \tau_2 = 0$, $k > 2$, while $\tau_2 \sigma^2$, $\sigma^2 \tau_2$, $\tau_2 \tau_1$ and $\tau_1 \tau_2$ contain only w_1, or σ^{n-3}, and $n - 3 > 2$. σ'^4 is accordingly linear in $\sigma^4 \ldots . \sigma^{n-3}$, σ'^5 is linear in $\sigma^5 \ldots . \sigma^{n-3}$, and so on up to σ'^{n-3}, which will contain only σ^{n-3}. That is, w_1' is the same as w_1, except for a constant factor. This theorem is not true for $n < 6$.

The system is now in the following form, only those products being given which are not definitely known.

	w_{n-4}	w_{n-3}	τ_1	τ_2
w_{n-4}			0	$\frac{a}{2}(3c + d) w_1$
w_{n-3}			aw_1	$c\tau_1 + \frac{a}{2}(c + d) w_2 + ew_1$
τ_1	0	$-aw_1$	bw_1	fw_1
τ_2	$-\frac{a}{2}(3d + c) w_1$	$d\tau_1 - \frac{a}{2}(c+d) w_2 - ew_1$	gw_1	$j\tau_1 + iw_2 + hw_1$

The remaining facts yielded by the associative law and the characteristic equation are mostly conditional, and subdivide the tables into classes. If $n > 6$ we have the following forms, reciprocal systems being considered equivalent:

I. If $b \neq 0$. Then $c = d = i = j = 0$.
II. If $b = 0$, $f^2 \neq g^2$. Then $c = d = i = j = 0$.
III. If $b = 0$, $f = g = a = 0$. Then $i = 0$.
IV. If $b = 0$, $f = g = 0$, $a \neq 0$. Then $i = j = 0$.
V. If $b = 0$, $f = g \neq 0$, $a = 0$. Then $c = d$, $i = cf$.
VI. If $b = 0$, $f = g \neq 0$, $a \neq 0$. Then $c = d$, $j = 0$, $i = cf$.
VII. If $b = 0$, $f = -g \neq 0$. Then $d = -c$, $i = cf$, $j = 0$.

By the transformation $\tau_2' = \tau_2 + \alpha\tau_1$, h can be made zero in I, II, V and VI, and e can be made zero in IV. With these simplifications the preceding forms will be called typical forms.

The case $n = 6$ presents some especial difficulties, and its details will not be considered in this paper. For the writer's present purposes it will suffice to state that if w_2 does not enter into τ_2^2, $\tau_2\sigma$ or $\sigma\tau_2$, the system can be put into one of the following typical forms:

I. $b \neq 0;\ j = c = d = h = 0.$
II. $b = 0,\ a \neq 0,\ c = 0;\ d = e = j = 0.$
III. $b = 0,\ a \neq 0,\ c \neq 0;\ d = -c,\ j = f = g = e = 0.$
IV. $b = 0,\ a = 0,\ c \neq 0;\ f = g = 0.$
V. $b = 0,\ a = 0,\ c = d = 0,\ f \neq g;\ j = 0.$
VI. $b = 0,\ a = 0,\ c = d = 0,\ f = g.$

The cases $n = 5$ and $n = 4$ are very simple, and the following may be put as typical forms:

$n = 5.$

	w_1	w_2	τ_1	τ_2
w_1	0	0	0	0
w_2	0	w_1	aw_1	$c\tau_1 + ew_1$
τ_1	0	$-aw_1$	bw_1	fw_1
τ_2	0	$d\tau_1 - ew_1$	gw_1	$j\tau_1 + hw_1$

I. $b \neq 0;\ c = d = j = h = 0.$
II. $b = 0,\ a \neq 0;\ c = d = e = j = 0.$
III. $b = a = 0,\ f \neq 0;\ c = d = j = 0.$
IV. $b = a = f = 0;\ g = 0.$

For $n = 4$, in the preceding table w_2 does not exist, and $b = j = h = 0$, $g = -f$.

A quantity α such that $\alpha x = -x\alpha$ for all values of x is called an alternate. Such cannot exist in a system with a modulus. A nilfactor is thus also an alternate. By actual trial in each of the given typical forms the following theorems can be demonstrated:

I. *A system in a typical form possesses no nilfactors except linear functions of* w_1 *and such* τ*'s as are themselves nilfactors.*

II. *A system in a typical form possesses no alternates except linear functions of* w_1 *and such* τ*'s as are themselves alternates.*

§6.

We will next proceed to reduce the parameters so far as possible for $n > 6$. By means of the transformation $\tau_1' = \tau_1 + \alpha\tau_2$, or by an interchange of the τ's, or both, Case I can be reduced to Case II if $f^2 \neq g^2$, to Cases V or VI (according as $e = 0$ or $e \neq 0$) if $f = g \neq 0$, to Case VII if $f = -g \neq 0$, and to Cases III or IV (according as $e = 0$ or $e \neq 0$) if $f = g = 0$.

Transformations $\tau_2' = x\tau_2$, $\tau_1' = y\tau_1$, $\sigma' = z\sigma$ enable us to reduce a great many parameters to unity or zero, dividing each typical form into a large number of subcases. Certain transformations show some of these subcases to be equivalent to others of the same group. Thus in III are employed

$$\sigma' = \sigma + \alpha\tau_2 \text{ and } \tau_1' = \tau_1 + \alpha w_1,$$

and in VII,

$$\tau_2' = \tau_2 + \alpha\tau_1, \; \sigma' = \sigma + \alpha\sigma^2, \text{ and } \sigma' = \sigma + \alpha\tau_1.$$

Furthermore, an interchange of the τ's shows an equivalence between certain cases of different groups.

There is thus found a total of 40 systems. To test directly to see what of these are linearly independent might require 780 applications of the general linear transformation. As a matter of fact it would require at least 364. The process is greatly reduced by the following considerations:

Suppose two systems $w_1 \ldots w_{n-3}, \tau_1, \tau_2, \eta$, and $w_1' \ldots w_{n-3}', \tau_1', \tau_2', \eta'$ are equivalent. Evidently $\eta = \eta'$. Consider any unit u' of the second system different from η'.

$$u' = a_1 w_1 + \ldots . a_{n-3} w_{n-3} + b_1\tau_1 + b_2\tau_2 + c\eta.$$

But c must be zero, else $u'^{n-2} \neq 0$. Hence, the incomplete systems $w_1 \ldots . w_{n-3}, \tau_1, \tau_2$ and $w_1' \ldots . w_{n-3}', \tau_1', \tau_2'$ are equivalent. We therefore need test only the incomplete systems.

First, the systems can be divided according as they are commutative or non-commutative. Secondly, since the number of linearly independent nilfactors is evidently a characteristic of the incomplete system, by the theorem on p. 382 these two groups can be divided according as none, one, or two of the τ's are nilfactors, the last case of which can occur, of course, only in the commutative class. Thirdly, since the number of linearly independent alternates is evidently a characteristic of the incomplete system, the subgroups of the non-commutative class can be subdivided according as none, one, or two of the τ's are alternates. These considerations separate the systems into eight distinct classes.

Next, supposing two systems to be equivalent, delete w_1. w_1 being unique and a nilfactor, the deleted systems must be equivalent by the theorem on page 377. The deleted systems will be in the typical w-τ forms with w_2 taking the place of w_1, and will be of deficiency 2 with one less unit. We can, therefore, subdivide each class according to the commutative, nilfactive, or alternate properties of the τ's with w_1 deleted.

Now, if two systems, with w_1 deleted, are equivalent, w_2 is unique, for it takes the place of w_1, and, since $n > 6$, the deleted system has at least six units (counting η). This does not imply that w_2 was originally equal to w_2', for they may have differed by the deleted w_1. Therefore, w_2 being a nilfactor, if we delete it, the deleted systems must be equivalent. The deleted systems will be in the typical w-τ forms with w_3 taking the place of w_1, and will still be of deficiency 2. We can, therefore, make subdivisions according to the commutative, nilfactive, or alternate properties of the τ's with w_1 and w_2 deleted. In some of the classes w_2 does not enter into any products, and in fact the application of this principle gives only three separations.

We can proceed thus to delete w's until we have only the four units w_{n-4}, w_{n-3}, τ_1, τ_2, although the deletions after w_2 change no multiplicative property of the τ's. w_{n-4} is not necessarily unique in the deleted systems (see theorem, p. 379), so it cannot be deleted. Now, if originally two systems were equivalent, they will be equivalent thus deleted. We can, consequently, apply the general linear transformation in four units to test the equivalence of the systems. The results, in common with those given in the last few paragraphs, are only negative; that is, equivalence in the deleted systems does not imply equivalence in the original systems, although non-equivalence necessitates non-equivalence originally.

Application of the general linear transformation in w_{n-4}, w_{n-3}, τ_1, τ_2 shows that a system in which $j=0$ is distinct from one in which $j \neq 0$, and a system in which $j=c=0$ or $j=d=0$ is distinct from one in which d, $c \neq 0$. These lead to further classifications and divide the systems into twenty-seven distinct groups, necessitating at most fifty applications of the general linear transformation to test equivalence.

The general linear transformation is

$$\begin{aligned} w'_{n-3} \equiv \sigma' &= a_1\sigma + a_2\sigma^2 + \ldots . a_{n-3}\sigma^{n-3} + b_1\tau_1 + b_2\tau_2, \\ \tau'_1 &= A_1\sigma + A_2\sigma^2 + \ldots . A_{n-3}\sigma^{n-3} + B_1\tau_1 + B_2\tau_2, \\ \tau'_2 &= \alpha_1\sigma + \alpha_2\sigma^2 + \ldots . \alpha_{n-3}\sigma^{n-3} + \beta_1\tau_1 + \beta_2\tau_2. \end{aligned}$$

$a_1 \neq 0$, else $\sigma'^{n-3} = 0$. $w'_{n-4} \ldots . w'_1$ follow as powers of σ'. From the general multiplicative properties of the system w'_{n-4} can contain only $\sigma^2 \ldots . \sigma^{n-3}$, τ_1, and $w'_{n-(4+k)}$ only $\sigma^{2+k} \ldots . \sigma^{n-3}$, and in each case the first term must occur, its coefficient being a_1^{2+k}. In fact

$$w'_1 = a_1^{n-3}w_1 \qquad w'_2 = a_1^{n-4}w_2 + bw_1 \qquad w'_3 = a_1^{n-5}w_3 + cw_2 + dw_1 + e\tau_1$$

where $e=0$ unless $n=7$, and b, c, d and e are independent of $a_4 \ldots . a_{n-3}$.

The products of τ'_1, τ'_2 with $\sigma'^{n-4} \ldots \sigma'^3$ show successively that $A_1, A_2 \ldots A_{n-6}$, $\alpha_1, \alpha_2 \ldots \alpha_{n-6}$ are all zero, and these conditions suffice to make the whole w'-τ' table of proper form except the products of τ'_1 and τ'_2 with themselves, each other, and with σ' and σ'^2. Changing the notation, we have accordingly

$$\begin{aligned} \sigma' &= a_1\sigma + a_2\sigma^2 + \ldots . a_{n-3}\sigma^{n-3} + b_1\tau_1 + b_2\tau_2, \\ \tau'_1 &= A_1w_1 + A_2w_2 + A_3w_3 + B_1\tau_1 + B_2\tau_2, \\ \tau'_2 &= \alpha_1w_1 + \alpha_2w_2 + \alpha_3w_3 + \beta_1\tau_1 + \beta_2\tau_2. \end{aligned}$$

Now, the products of τ'_1 and τ'_2 with σ', $\sigma'^2 \ldots . \sigma'^{n-3}$ expressed in w's and τ's, are not affected by the values of $a_4 \ldots . a_{n-3}$. It follows from the two preceding sets of equations that the values of w_1, w_2, w_3, τ_1 and τ_2 expressed in w'_1, w'_2, w'_3, τ'_1, τ'_2 are independent of $a_4 \ldots . a_{n-3}$. Therefore the multiplication table for $w'_1 \ldots . w'_{n-3}$, τ'_1, τ'_2 is unaffected by making $a_4 = a_5 = \ldots . a_{n-3} = 0$. Thus the transformations are simplified.

By these transformations the writer has tested the various forms for equivalence and non-equivalence, and has obtained the following table which comprises all the possible non-equivalent forms, provided any system is always regarded as the equivalent of its reciprocal, the letters having the signification given in the form on p. 380:

a	b	c	d	e	f	g	h	i	j
0	0	0	0	0	1	1	0	0	1
0	0	1	1	0	1	1	0	1	0
0	0	0	0	0	0	0	1	0	0
0	0	0	0	0	0	0	0	0	1
0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	1	g	0	0	0
1	0	0	0	1	1	1	0	0	0
1	0	0	0	0	1	1	0	0	0
0	0	0	0	0	1	-1	1	0	0
0	0	0	0	1	1	1	0	0	1
0	0	1	1	1	1	1	0	1	0
1	0	1	1	e	1	1	0	1	0
1	0	1	-1	0	1	-1	1	1	0
0	0	1	-1	0	1	-1	0	1	0
1	0	c	$2-c$	0	0	0	h	0	0
1	0	1	-1	0	1	-1	0	1	0
1	0	1	-1	0	0	0	h	0	0
1	0	0	0	0	0	0	1	0	0
0	0	1	d	0	0	0	1	0	0
0	0	1	d	0	0	0	0	0	0
0	0	1	0	0	0	0	1	0	1
0	0	0	1	0	0	0	0	0	1
0	0	0	0	1	0	0	0	0	1
0	0	1	1	1	0	0	0	0	0
0	0	1	1	1	0	0	1	0	0
0	0	0	0	1	0	0	1	0	0
1	0	0	0	0	0	0	0	0	0

To these must be added, if we do not regard reciprocals as necessarily equivalent:

a	b	c	d	e	f	g	h	i	j
0	0	0	1	0	0	0	1	0	0
0	0	0	1	0	0	0	0	0	0

In one of the systems e enters as a parameter. It is a little remarkable that it can be made zero except for the particular case $n = 8$, the necessary transformations containing a fraction with $(n - 8)$ in the denominator. Similarly, in the two systems in which h enters as a parameter, it can be reduced to unity if it is not zero, except for the particular case $n = 7$.

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